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RESEARCH ARTICLE

A general solution for the receding contact problem of a functionally graded layer resting on a Winkler foundation

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Abstract

In this paper, the receding contact problem of functionally graded (FG) layer resting on a Winkler foundation is considered. It is assumed that the shear modulus of the layer change functionally along the depth whereas Poisson ratio remains constant. Arbitrary concentrated loads by means of arbitrary rigid punches are applied to the top of the layer. The problem is considered as a plain strain problem. A general formulation is obtained using elasticity theory and Fourier integral transform. Obtained formulation is valid for both symmetric and asymmetric systems. A parametric study is carried out to investigate the effect of material properties and loading on contact distances and contact pressures. It is found that, increasing rigidity of the bottom of the FG layer compared to the top of the FG layer, the contact distances between the circular punch and FG layer contact surface decreases whereas maximum contact pressure increases. In addition, placement of the rigid punches has an effect on the contact distances and contact pressures.

Keywords

Receding contact, Elasticity; Functionally graded material; Fourier transform; Receding contact

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1. Introduction

There are many engineering applications where the stress analysis at the interface between two bodies in contact is principal in the structural design as the response of the structure depends on it. Examples of these applications in engineering are railways, foundation grillages, connecting rods, joint and support elements, rolling mills, pavements of highway and airfield etc. [1,2]. So, problems involving the contact of two separate bodies pressed against each other have been widely studied by many researchers. Although the contact area increases after the application of the load in many cases, there are others where the contact area becomes smaller. This kind of problem is called receding in literature.

Among the analytical studies on receding contact, the followings are recorded in literature. Keer et.al. [3] solved the smooth receding contact problem between an elastic layer and a half space when two bodies were pressed against each other by considering both plane and axisymmetric cases. The frictionless contact problem for an elastic layer resting on two quarter planes and loaded compressively was solved by Erdogan and Ratwani [4]. Civelek and Erdogan [5] investigated the general axisymmetric double frictionless contact problem for an elastic layer resting on a half space and pressed by an elastic stamp. The smooth receding contact problem for an elastic layer

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pressed against a half space by frictionless semi-infinite elastic was examined by Gecit [9]. Akavcı [6] studied a contact problem for an elastic layer supported by two elastic quarter planes both symmetrical and axisymmetric loadings. Comez et.al. [7] solved double receding contact problem for two elastic layers having different elastic constants and heights and pressed by a rigid stamp. Kahya et.al. [8] considered a frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half plane, when the two bodies were pressed together by means of a rigid circular stamp. Yaylacı and Birinci [9] studied a receding contact problem of two elastic layers supported by two elastic quarter planes. The solution of a receding contact problem using an analytical method and a finite element method was examined by Oner et.al. [10]. Karabulut et. al. [11] studied a receding contact problem of layer resting on a half plane and pressed by two flat blocks placed symmetrically

The processing techniques related to functionally graded materials (FGMs) and the importance of these processing was considered by Kieback et. al. [12]. Jin and Batra [13] studied the thermal stresses and the stress intensity factor in an edge-cracked strip of a functionally graded material (FGM) subjected to sudden cooling at the cracked surface. The geometrically nonlinear response of inhomogeneous isotropic and functionally graded plates and shells was considered by Hui-Shen [14]. Sofiyev [15] focused on the thermal buckling of FGM shells resting on a two-parameter elastic foundation. The buckling of cylindrical shells made of FGM in contact with the Pasternak elastic foundation subjected to uniform temperature rise is investigated by Bagherizadeh et. al. [16]. Tornabene [17] investigated recovery of through-the-thickness transverse normal and shear strains and stresses in statically deformed functionally graded (FG) doubly-curved sandwich shell structures and shells of revolution using the generalized zigzag displacement field and the Carrera Unified Formulation.

A receding contact plane problem for a functionally graded layer pressed against a homogeneous half space was analyzed by El-Borgi et.al. [18]. A multi-layered model for sliding frictional contact analysis of functionally graded materials (FGMs) with arbitrarily varying shear modulus under plane strain-state deformation has been developed by Ke and Wang [19]. The two-dimensional frictionless contact problem of a coating structure consisting of a surface coating, a functionally graded layer and a substrate under a rigid cylindrical punch was investigated by Yang and Ke [20]. Barik et.al. [21] studied the stationary plane contact of a functionally graded heat conducting punch and a rigid insulated half-space. The frictionless contact problem of a functionally graded piezoelectric layered half-plane in-plane strain state under the action of a rigid flat or cylindrical punch was examined by Ke et.al. [22]. Sliding frictional contact between a rigid punch and a laterally graded elastic medium was studied by Dag et.al.[23]. Rhimi et.al. [24,25] considered the axisymmetric problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space when the two bodies were pressed together and double receding contact between a rigid stamp of axisymmetric profile, an elastic functionally graded layer and a homogeneous half space. Chen and Chen [26] studied the contact behaviors of a graded layer resting on a homogeneous half space and pressed by a rigid stamp. Comez [27] considered a contact problem for a functionally graded layer loaded by means of a rigid stamp and supported by a Winkler foundation. The plane problem of a frictional receding contact formed between an elastic functionally graded layer and a homogeneous half space, when they were pressed against each other, was investigated by El-Borgi et.al. [28]. Adiyaman et. al. [29] studied the receding contact problem of FG layer resting on quarter planes and loaded by a symmetrically places distributed load. The double receding contact problem between a FG layer and a homogeneous layer investigated by Yan and Li [30]. Liu et. al. [31] studied the axisymmetric receding contact problem of a FG coating under a rigid circular block. The axisymmetric contact problem of a FG layer resting on an elastic substrate investigated by Turan et. al. [32]. Yan and Mi [33,34] studied contact problems of a structure consisted of a FG layer, a homogeneous layer and homogenous half plane.

Although the contact problem of a layer resting on Winkler foundation has been studied [35,27], the problems were solved only for a specific symmetric loading case. However, in this study, a general solution valid for any loading case, whether it is symmetric or asymmetric, is derived. In addition, this solution is compatible for programming purposes. Therefore, a computer program with a graphic user interface is developed and the numerical results are obtained used this program. Obtained numerical results are given in tables and figures.

2. Definition of the problem

The general solution of the contact problem of a FG layer of thickness h resting on a Winkler foundation is considered. It is assumed that the shear modulus of the layer, G, changes exponentially thorough the depth as given below whereas the Poisson ratio, υ, remains constant.

$$G = G_0 e^{\beta y} \tag{1}$$

In which, G_0 is the shear modulus of the layer at the bottom ($y = 0$) and β is the non-homogeneity parameter which represents the change in the shear modulus. The loading and the geometry of the problem is given in Fig. 1 as representatively. The layer is loaded with n concentrated loads by means of arbitrary rigid blocks. It is assumed that the layer is attached to the foundation and the effect of the gravity is neglected. The problem is considered as a plain strain problem.

3. Formulation of the problem

The equilibrium equations in terms of displacements (Navier Equations) for a FG layer can be obtained as follows.

$$\begin{aligned}(\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} \\ +\beta(\kappa-1) + B(\kappa-1)\frac{\partial v}{\partial x} = 0\end{aligned} \tag{2}$$

$$\begin{aligned}(\kappa-1)\frac{\partial^2 v}{\partial x^2} + (\kappa+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} \\ +\beta(3-\kappa)\frac{\partial u}{\partial x} + \beta(\kappa+1)\frac{\partial v}{\partial y} = 0\end{aligned} \tag{3}$$

in which, u and v are the x and y components of the displacement field, respectively; σ_x, σ_y and τ_{xy} are the components of the stress field in the same coordinate system; ε_x, ε_y and ε_{xy} are the corresponding components of the strain field; and κ is a material property defined as $\kappa = 3-4\nu$ for plane strain problems. Eqs. (3, 4) are subjected to following boundary conditions.

$$\sigma_y(x,0) = -kv(x,0) \quad -\infty < x < \infty \tag{4}$$

$$\tau_{y_1}(x,0) = \tau_{y_1}(x,h) = 0 \quad -\infty < x < \infty \tag{5,6}$$

$$\sigma_{y_N}(x,h_N) = \begin{cases} -p_1(x) & b_{1L} < x < b_{1R} \\ \vdots & \vdots \\ -p_j(x), & b_{jL} < x < b_{jR} \\ \vdots & \vdots \\ -p_n(x) & b_{nL} < x < b_{nR} \end{cases} \tag{5}$$

$$\frac{\partial v_j(x,h)}{\partial x} = f_j \quad -b_{jL} < x < b_{jR},\ j = 1,2,...,n \tag{6}$$

In these boundary conditions, k is the stiffness parameter of the Winkler foundation, p_j ($j = 1,2,...,n$) represents unknown contact pressures under the j^{th} block; b_{jL} and b_{jR} are unknown starting and ending points of the contact, respectively; and f_j is the derivative of the shape function of the j^{th} block with respect to x. Equilibrium conditions for the problem can be written as follows;

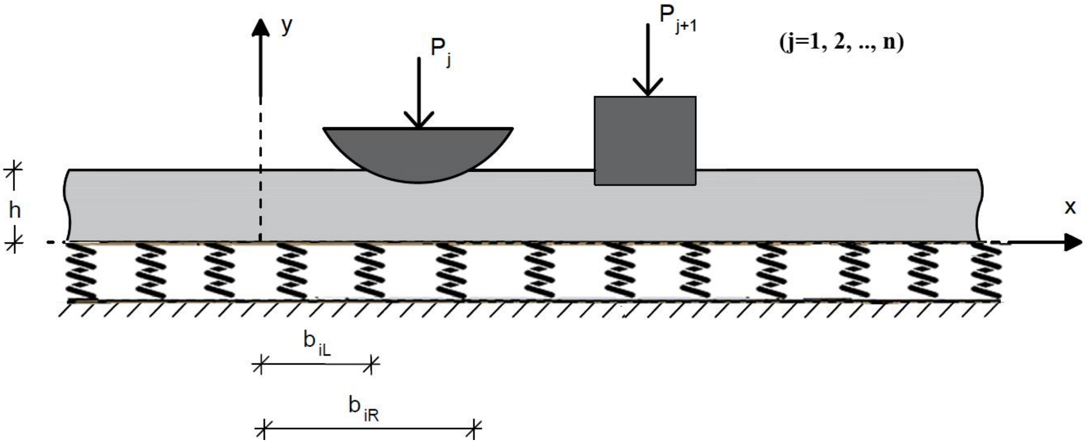

Fig. 1. The geometry and loading of the problem

$$\int_{b_{jL}}^{b_{jR}} p_j(x)dx = P_j \qquad (j = 1,2,...,n) \qquad (7)$$

in which, P_j $(j = 1,2,...,n)$ is the concentrated load applied to top of the j^{th} block.

4. Solution of the problem

Using Fourier integral transform, u and v can be written as follows;

$$u(x,y) = \int_{-\infty}^{\infty} \phi(\xi,y)e^{-i\xi x}dx, \qquad (8)$$

$$v(x,y) = \int_{-\infty}^{\infty} \psi(\xi,y)e^{-i\xi x}dx, \qquad (9)$$

in which, $\phi(\xi,y)$ and $\psi(\xi,y)$ are the Fourier transform of u and v with respect to the x-coordinate, respectively. Plane elasticity equations (2,3) are partial differential equations and can be converted into ordinary differential equations using Fourier integral transform. The characteristic equation associated with these ordinary differential equations can be obtained as follows.

$$n_j^4 + 2\beta n_j^3 + (\beta^2 - 2\xi^2)n_j^2 - 2\xi^2\beta n_j + \xi^2(\xi^2 + \beta^2\frac{3-\kappa_1}{\kappa_1+1}) = 0 \qquad (10)$$

The roots of Eq. (10) can be expressed as given.

$$n_1 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa}{\kappa+1}}}\right) \qquad (11)$$

$$n_2 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa}{\kappa+1}}}\right) \qquad (12)$$

$$n_3 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa}{\kappa+1}}}\right) \qquad (13)$$

$$n_4 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa}{\kappa+1}}}\right) \qquad (14)$$

Solving ordinary differential equation represented by (12), the displacement and stress components of the FG layer can be expressed as follows.

$$u(x,y) = \int_0^{\infty}\sum_{k=1}^{4}\left(A_k e^{n_k y}\right)e^{-i\xi x}d\xi \qquad (15)$$

$$v(x,y) = \int_0^{\infty}\sum_{k=1}^{4}\left(m_k A_k e^{n_k y}\right)e^{-i\xi x}d\xi \qquad (16)$$

$$\sigma_y(x,y) = \frac{G(y)}{K-1}\int_0^{\infty}\sum_{k=1}^{4}\left(C_k A_k e^{n_k y}\right)e^{-i\xi x}d\xi \qquad (17)$$

$$\tau_{xy}(x,y) = G(y)\int_0^\infty \sum_{k=1}^{4}\left(D_k A_k e^{n_k y}\right)e^{-i\xi x}d\xi \qquad (18)$$

In these components; m_k, C_k and D_k ($k = 1,...,4$) are known functions whereas A_k are unknown coefficient functions. Applying boundary conditions (4-7), unknown coefficients can be obtained in terms of unknown contact pressures, p_j ($j = 1,2,...,n$), and contact distances, b_{jL} and b_{jR}.

Using unused boundary condition given in Eq. (8), the solution of the problem can be converted into the solution of an integral equation system which consists of n singular integral equations and unknown contact pressures and contact distances can be obtained using the solution method suggested in [36] from the solution of this system.

Since the problem involves arbitrary loading, the analytical solution of the problem should be carried out for each loading case. In order to achieve this, a computer program is coded such that it performs the analytical solution of the problem using suggested method and displays obtained results.

5. Numerical results and discussion

The height of the graded layer, h, is taken as 1 whereas the Poison's ratio of the graded layer is taken as 0,25. Note that all quantities in the tables and figures are normalized. G_h defined as the shear modulus at the top of the FG layer ($y = h$).

$$G_h = G_0 e^{\beta h} \qquad (19)$$

As the first loading case, a symmetric problem from the literature [27] is chosen in order to compare the results. In this problem, the layer is subjected to a concentrated load, P, by means of a circular rigid block with radius, R, placed at the symmetry axis and is resting on a Winkler foundation. The contact distances occurs in $[-a, a]$ in [27] whereas the contact distances occurs in [b_{1L}, b_{1R}] in this study. Since the problem is symmetric with respect to geometry as well as loading, comparing only a and b_{1R} values can be sufficient.

Table 1 shows the comparison of a and b_{1R} for various G_0/G_h and k/G_h ratios. It can be seen that increasing G_0/G_h and k/G_h results a reduction in contact distances. In addition, the written program produces almost same values in the literature.
The contact pressures for various G_0/G_h ratios are given in Fig. 2. As it is seen from the figure, if the G_0/G_h ratio increases, contact distances decreases. Moreover, contact pressures increases for decreasing contact distances. Maximum pressures occur under the middle of the circular block.

As the second loading case, a symmetric problem with two identical circular blocks is chosen. The blocks are placed such that the distances from the middle of the blocks to the y axis is same and loaded with same concentrated loads. The first block is at the left whereas the second block is at the right.

Table 2 shows the contact distances for various block positions. As it can be seen from the table, since the problem is symmetric, obtained contact distances for one block are compatible with the contact distances of the other block. In addition, when blocks approach to each other, b_{1R} and b_{2L} values approach to the middle of the block whereas b_{1L} and b_{2R} values move away from the middle of the block.

The change in contact pressures under the left block and right block for various block positions is given in Fig. 3 and 4, respectively. It can be seen from the figures, maximum contact pressures increases if the block approaches to each other. Also, the pressures under the block are comparable because of symmetry.

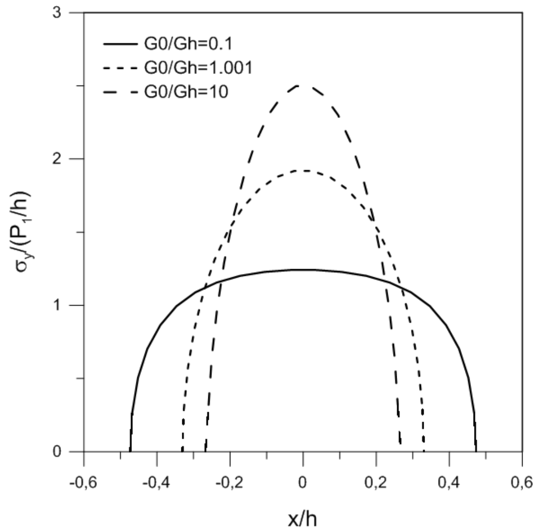

Fig. 2. Dimensionless contact pressures for various G_0/G_h ratios in case of one block

Table 1. The comparison of half contact distances for various G_0/G_h and k/G_h ratios in case of one block ($R/h = 100$)

$\frac{k}{G_h}$	$\frac{G_0}{G_h} = 0.1$		$\frac{G_0}{G_h} = 1.001$		$\frac{G_0}{G_h} = 10$	
	Comez [27]	Present study	Comez [27]	Present study	Comez [27]	Present study
0.1	0.6220	0.6222	0.3725	0.3724	0.2779	0.2780
1	0.4734	0.4732	0.3304	0.3304	0.2673	0.2674
10	0.4452	0.4451	0.3151	0.3150	0.2617	0.2618
$\to \infty$	0.4415	0.4420	0.3116	0.3115	0.2594	0.2595

Table 2. The contact distances for various block positions in case of two symmetrical blocks ($G_0/G_h = 0.5$, $P_1/P_2 = 1$, $k/G_h = 0.1$, $R_1/h = 100$)

$\lvert x/h \rvert$	b_{1L}	b_{1R}	b_{2L}	b_{2R}
5	-5.240230	-4.759518	4.759518	5.240230
1	-1.240149	-0.761891	0.761891	1.240149
0.5	-0.758593	-0.290634	0.290634	0.758593
0.25	-0.535542	-0.083235	0.083235	0.535542

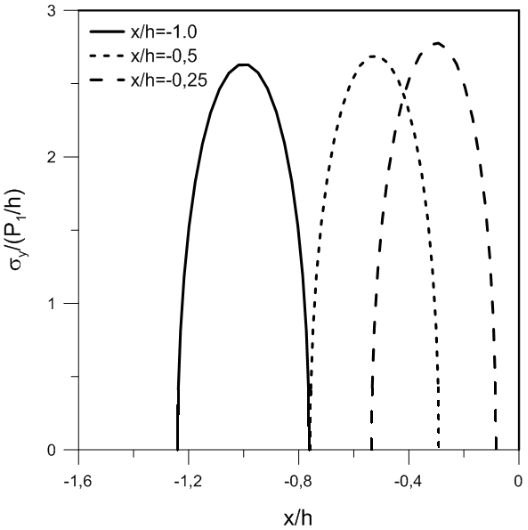

Fig. 3. Dimensionless contact pressures under the left block for various block positions in case of two symmetric blocks

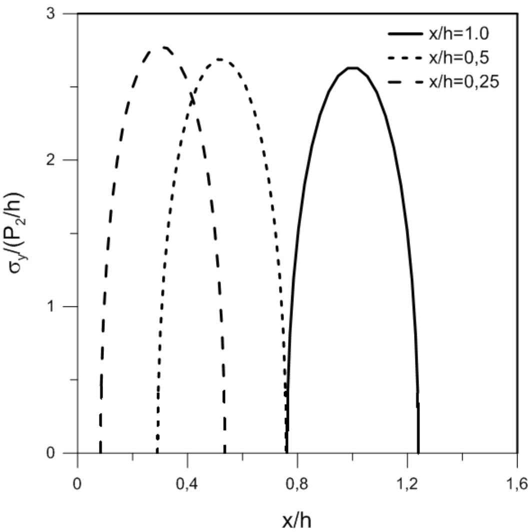

Fig. 4. Dimensionless contact pressures under the right block for various block positions in case of two symmetric blocks

As the last loading case, a problem with three blocks is chosen. The left and right blocks are cylindrical with similar or different radius. Whereas the block in the middle is flat. First, second and third blocks are at the left, in the middle and at the right, respectively.

The variation of contact distances for various block radius ratios is given in Table 3. The radius of the right block is kept constant whereas the radius of the left block is changed. It can be seen from the table, the contact distances are not comparable between left and right blocks in case of asymmetric loading ($R_1 / R_3 \neq 1$). In addition, the start and end of the contact distances goes to left and right, respectively, if R_1 / R_3 increases. In other words, the contact distance increases because of the increase in the block radius. Although, the block radius of the right block is kept constant, there are small changes in the contact distances. Similar to left block, the start and end of the contact distances goes to left and right, respectively, and the contact distance increases if R_1 / R_3 increases.

Figs. 5-7 show the change in the contact pressures for left block, right block and middle block, respectively, in case of various block radius ratios. As it is seen from the figures, if the radius of the block increases, the maximum contact pressure under the block decreases. Increasing the radius of the left block does not much affect in the contact pressure of the other two punches whether it is circular or flat and the contact pressure graphs for various block radius ratios are overlapped. Compared to circular blocks, the pressures under the flat block goes to infinite at the ends of the block.

Table 3. The contact distances for various block radius ratios in case of three asymmetrical blocks ($G_0 / G_h = 0.5$, $P_1 / P_2 = 1$, $P_1 / P_3 = 1$, $k / G_h = 0.1$, $R_3 / h = 100$)

R_1 / R_3	b_{1L}	b_{1R}	b_{3L}	b_{3R}
0,25	-2.117883	-1.884599	1.766879	2.243265
0,5	-2.169165	-1.835812	1.766802	2.243333
1,0	-2.243361	-1.766792	1.766792	2.243361
2,0	-2.350717	-1.671286	1.766743	2.243644

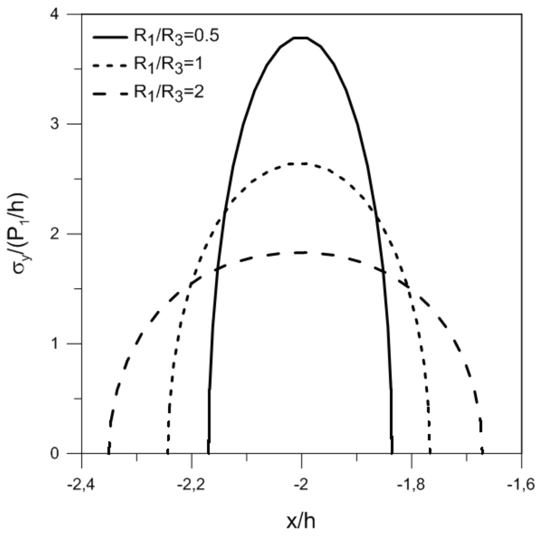

Fig. 5. Dimensionless contact pressures under the left block for various block radius ratios in case of three asymmetric blocks

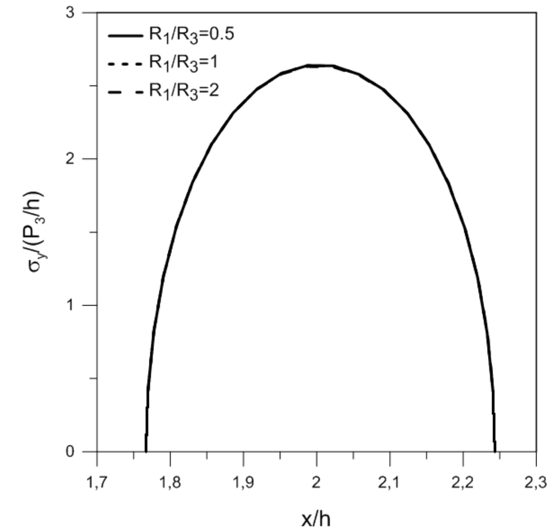

Fig. 6. Dimensionless contact pressures under the right block for various block radius ratios in case of three asymmetric blocks

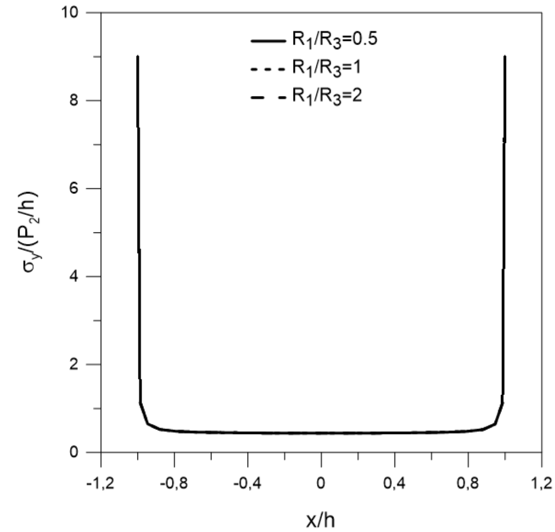

Fig. 7. Dimensionless contact pressures under the middle block for various block radius ratios in case of three asymmetric blocks

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